Rotating charged dust in general relativity

1980 J. Phys. A: Math. Gen. 13 3465

(http://iopscience.iop.org/0305-4470/13/11/019)

J. Phys. A: Math. Gen. **13** (1980) 3465–3477. Printed in Great Britain

Rotating charged dust in general relativity

W B Bonnor

Mathematics Department, Queen Elizabeth College, University of London, Atkins Building, Campden Hill, London W8 7AH, UK

Received 14 April 1980

Abstract. The relativistic theory of axially symmetric, rotating charged dust is compared with the classical theory. The equations of motion are obtained from the field equations as compatibility conditions. It is proved that, if comoving coordinates are used for a rigid rotation in which the Lorentz force does not vanish, σ/ρ and g_{44} are functions of the electric potential. Two new classes of exact solutions are obtained, one being the general solution for the case of rigid rotation with vanishing Lorentz force.

1. Introduction

Early work on rotating charged dust (Som and Raychaudhuri 1968, Banerjee and Banerji 1968) described cylindrically symmetric space–times in which the Lorentz force vanishes; these included some interesting variants on the Gödel universe. At about the same time (De and Raychaudhuri 1968, Raychaudhuri and De 1970) some general theorems about flows of charged dust were discovered, and the electromagnetic version of the Raychaudhuri equation was formulated.

In 1976, Banerjee *et al* showed that, in the case of stationary charged dust with vanishing Lorentz force, it is possible, by the use of comoving coordinates, to reduce the electric field to zero and have $g_{44}=1$. It was also proved that, under a certain assumption, the ratio of charge to mass densities is an arbitrary constant, thereby correcting some previous work by other authors.

In a series of papers, Islam (1977, 1978, 1979) has considered axially symmetric, rotating, charged dust, both from the classical and relativistic points of view. He has also derived some exact solutions in both theories. In this paper I shall follow the approach of Islam, at one point verifying the set of field equations he gave, which may be helpful as they are complicated. I shall also generalise some of his solutions.

Space–times for rotating charged dust are interesting because they show the interplay of four fields—gravitation, rotation, electricity and magnetism. They also exhibit in an instructive way similarities and differences between the classical and relativistic theories.

In § 2 I develop the classical theory of charged dust and in § 3 I write down the field equations in the relativistic case. In § 4 the relativistic equations of motion are obtained from the field equations and compared with the classical ones obtained in § 2. § 5 sees the specialisation to rigid rotation, which is maintained in the remainder of the paper; with this specialisation the equations of motion simplify, and the space–times divide naturally into two classes according as the Lorentz force does not or does vanish. Sections 6 and 7 give some exact solutions in the relativistic theory, and there is a brief conclusion in § 8. The new results are to be found chiefly in §§ 4, 6 and 7.

0305-4470/80/113465+13\$01.50

2. Charged dust in Newton–Maxwell theory

The equations for rotating charged dust in Newton–Maxwell theory have been formulated and studied by Islam (1978). I shall rederive them briefly, as they will be useful for interpreting the relativistic results in §§ 4, 6, 7. The field will be assumed stationary, and symmetric about an axis $0z$; the constant of gravitation and the velocity of light will be taken as unity. Rectangular Cartesian coordinates will be used. All functions will, except where stated, be dependent on z and r $(=(x^2+y^2)^{1/2})$ only.

It will be assumed that the dielectric constant and the magnetic permeability are both unity, so that only $\boldsymbol{E}$ and $\boldsymbol{H}$ (and not $\boldsymbol{D}$ and $\boldsymbol{B}$) appear in the equations. I shall also suppose that there is no longitudinal component of the current present, so that the azimuthal component of the magnetic field $\boldsymbol{H}$ vanishes. Thus the components of $\boldsymbol{H}$ and the electric field $\boldsymbol{E}$ will be taken as

$$\boldsymbol{H}=(xr^{-1}\beta, yr^{-1}\beta, \gamma) \qquad \boldsymbol{E}=(xr^{-1}\chi, yr^{-1}\chi, \nu). \tag{2.1}$$

The velocity of the dust is

$$\boldsymbol{v}=(-y\Omega, x\Omega, 0), \tag{2.2}$$

and we shall suppose that the dust carries the charge, so the current will be taken as

$$\boldsymbol{J}=\sigma\boldsymbol{v} \tag{2.3}$$

where σ is the charge density.

The Maxwell equations for the stationary case are

$$\nabla\,.\,\boldsymbol{E}=4\pi\sigma \qquad \nabla\times\boldsymbol{E}=0 \tag{2.4}$$

$$\nabla\,.\,\boldsymbol{H}=0 \qquad \nabla\times\boldsymbol{H}=4\pi\boldsymbol{J}, \tag{2.5}$$

and the Newtonian gravitational potential satisfies

$$\nabla^2 V=4\pi\rho \tag{2.6}$$

where ρ is the mass density. The equation of motion is

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{v}\,.\,\nabla\right)\boldsymbol{v}=-\rho\nabla V+\sigma(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{H}). \tag{2.7}$$

From the second of equations (2.4) we may write

$$\boldsymbol{E}=-\nabla\phi \tag{2.8}$$

where ϕ is the electrostatic potential and satisfies

$$\nabla^2\phi=-4\pi\sigma \tag{2.9}$$

because of the first of equations (2.4). Equations (2.5) first give

$$r^{-1}\left(\frac{\partial}{\partial r}(r\beta)+\frac{\partial}{\partial z}(r\gamma)\right)=0, \tag{2.10}$$

and secondly, with the use of (2.3),

$$\frac{\partial\gamma}{\partial r}-\frac{\partial\beta}{\partial z}=-4\pi r\sigma\Omega. \tag{2.11}$$

Because of (2.10) we may introduce a function ψ by

$$r\beta = \partial\psi/\partial z \qquad r\gamma = -\partial\psi/\partial r, \tag{2.12}$$

whence, from (2.11)

$$\nabla^{*2}\psi = 4\pi r^2\sigma\Omega \tag{2.13}$$

where $\nabla^{*2} \equiv (\partial^2/\partial z^2) + (\partial^2/\partial r^2) - (1/r)\partial/\partial r$. It follows that the gravitational, electric and magnetic fields are generated by the three potentials V, ϕ and ψ satisfying (2.6), (2.9) and (2.13) respectively.

It remains to be seen how these fields are restricted by the equation of motion (2.7). Using (2.1), (2.2), (2.8) and (2.12) we find

$$\rho(V - \tfrac{1}{2}\Omega^2 r^2)_a + \sigma(\phi + \Omega\psi)_a + (r^2\rho\Omega - \sigma\psi)\Omega_a = 0 \tag{2.14}$$

where a suffix a means differentiation with respect to z or r. We shall be concerned in the rest of this section with rigid rotations, i.e. with Ω = constant ($\neq 0$). Since we are interested in the motion of charged dust, *we shall suppose* $\rho \neq 0$, $\sigma \neq 0$ *except possibly at isolated points.*

It is convenient to divide the treatment of (2.14) into two cases, I and II, according as $\phi + \Omega\psi$ is not, or is constant.

(*a*) *Case I.* $\phi + \Omega\psi \neq$ constant:

$$V - \tfrac{1}{2}\Omega^2 r^2 = H(\phi + \Omega\psi) \qquad \text{and} \qquad \rho H' + \sigma = 0 \tag{2.15}$$

where H is a (non-constant) function of $\phi + \Omega\psi$ and a prime means differentiation with respect to this argument. Of particular interest is the sub-case in which H is a linear function of its argument.

(*b*) *Case I*(*a*).

$$V - \tfrac{1}{2}\Omega^2 r^2 = n(\phi + \Omega\psi) \tag{2.16a}$$

$$n\rho + \sigma = 0 \tag{2.16b}$$

where n is a constant so that the ratio of the charge to mass density σ/ρ is constant. Taking the Laplacian of (2.16*a*), and using (2.6), (2.9), (2.13) and (2.16*b*), we get

$$\sigma = \frac{n\Omega^2(1 + r^{-1}\Omega^{-1}n\psi_r)}{2\pi(n^2 - 1 - n^2r^2\Omega^2)} \tag{2.17}$$

where ψ_r means $\partial\psi/\partial r$. It is convenient to introduce a new magnetic potential

$$\alpha := \psi + \tfrac{1}{2}n^{-1}r^2\Omega \tag{2.18}$$

(so that α satisfies (2.13) if ψ does) which reduces (2.17) to

$$\sigma = \frac{n^2\Omega\alpha_r}{2\pi r(n^2 - 1 - n^2r^2\Omega^2)}. \tag{2.19}$$

This corresponds physically to introducing a uniform magnetic field parallel to the z axis which gives a term in (2.14) balancing the centrifugal force. Reverting to (2.13) and substituting for ψ from (2.18) and for σ from (2.19) we obtain a differential equation for α:

$$\alpha_{zz} + \alpha_{rr} - \alpha_r \frac{n^2 - 1 + n^2r^2\Omega^2}{r(n^2 - 1 - n^2r^2\Omega^2)} = 0. \tag{2.20}$$

n and Ω are free parameters and once they are fixed, α obtained from (2.20) determines the entire solution for case I(a) provided suitable boundary conditions are specified on V and ϕ.

Notice that if $n^2 = 1$ (2.20) reduces to a Laplace equation, and the charge and mass densities are numerically equal. The purely electrostatic instance of this case is well known, but here the interpretation is less transparent because of the magnetic field; in particular the charge density has to satisfy a differential equation (obtained by eliminating α between (2.19) and (2.20)) and is not arbitrary as in the electrostatic case.

(c) *Case II. The Lorentz force vanishes*: i.e. the bracket in the second term on the right of (2.7) is zero. Since Ω is constant, this means that

$$\phi + \Omega\psi = \text{constant}, \tag{2.21}$$

so, from (2.14),

$$V = \tfrac{1}{2}\Omega^2 r^2 + V_0 \qquad (V_0 \text{ constant}) \tag{2.22}$$

and thence from (2.6)

$$2\pi\rho = \Omega^2. \tag{2.23}$$

Since the electromagnetic field exerts no force on the matter the centrifugal force must be balanced purely by gravitation, as expressed in (2.23). Taking the Laplacian of (2.21) and using (2.13) we obtain

$$\sigma = [2\pi r(1 - r^2\Omega^2)]^{-1}\Omega\psi_r, \tag{2.24}$$

and substituting for σ from (2.24) into (2.13) we find

$$\psi_{zz} + \psi_{rr} - \frac{\psi_r}{r}\frac{1 + r^2\Omega^2}{1 - r^2\Omega^2} = 0. \tag{2.25}$$

Given Ω, a solution ψ of this equation determines σ from (2.24) and ϕ from (2.21). ρ is obtained from (2.23), irrespective of ψ.

By way of illustration, and later comparison with a relativistic solution, we choose the simple solution of (2.25),

$$\psi = L \ln(1 - r^2\Omega^2) + M \tag{2.26}$$

where L and M are constants. This is the only one corresponding to a purely longitudinal, (but non-uniform) magnetic field. The charge density, obtained from (2.24), is given by

$$\sigma = \frac{-L\Omega^3}{\pi(1 - r^2\Omega^2)^2}. \tag{2.27}$$

The gravitational potential and mass density are given by (2.22) and (2.23). The solution has a singularity where $r^2\Omega^2 = 1$, that is, where the speed of the rotating dust becomes equal to that of light.

3. The equations for charged dust in general relativity

The Einstein–Maxwell equations for charged dust are

$$R^i_k - \tfrac{1}{2}\delta^i_k R = 8\pi\rho u^i u_k + 2F^{ia}F_{ka} - \tfrac{1}{2}\delta^i_k F^{ab}F_{ab} \tag{3.1}$$

$$F_{ik} = A_{i,k} - A_{k,i} \tag{3.2}$$

$$F^{ik}_{\ ;k} = 4\pi J^i \tag{3.3}$$

where R_{ik} denotes the Ricci tensor of the space–time with metric g_{ik}, defined by

$$R_{ik} = \Gamma^a_{ia,k} - \Gamma^a_{ik,a} + \Gamma^b_{ia}\Gamma^a_{kb} - \Gamma^a_{ik}\Gamma^b_{ab};$$

ρ is the matter density, u^i the unit four-velocity vector of the dust ($u^iu_i = 1$), F_{ik} the electromagnetic field tensor, A_i the vector potential and J^i the current density; a comma denotes partial differentiation and a semicolon covariant differentiation.

We suppose that the axially symmetric stationary metric is

$$ds^2 = -e^\mu(dz^2 + dr^2) - l d\theta^2 - 2m d\theta dt + f dt^2 \tag{3.4}$$

where μ, l, m and f are functions of z and r only. The coordinates will be numbered

$$x^1 = z \qquad x^2 = r \qquad x^3 = \theta \qquad x^4 = t. \tag{3.5}$$

I shall assume that the dust is steadily rotating, which means I shall take

$$u^i = (0, 0, u^3, u^4) \qquad A_i = (0, 0, \psi, \phi) \qquad J^i = (0, 0, J^3, J^4), \tag{3.6}$$

all these qualities being functions of z and r only. By doing this I exclude axially symmetric, stationary space–times which have motion and currents parallel to $0z$.

Let

$$\Delta^2 = lf + m^2; \tag{3.7}$$

then we find the field equations (3.1) that

$$R^3_3 + R^4_4 \equiv -\Delta^{-1} e^{-\mu}(\Delta_{11} + \Delta_{22}) = 0, \tag{3.8}$$

where suffices 1 and 2 mean $\partial/\partial x^1$ and $\partial/\partial x^2$. Assuming Δ to be a monotonically increasing function of r, we may, without loss of generality (Synge 1960), take the solution of (3.8) to be

$$\Delta = r \tag{3.9}$$

so that

$$lf + m^2 = r^2. \tag{3.10}$$

With Δ chosen as in (3.9), and with assumptions (3.6), the remaining equations (3.1) can be written in terms of the linear combinations

$$R_{11} + R_{22} \equiv \mu_{11} + \mu_{22} + \tfrac{1}{2}f^{-2}(f_1^2 + f_2^2) - r^{-1}f^{-1}f_2 - \tfrac{1}{2}r^{-2}f^2(w_1^2 + w_2^2) = -8\pi\, e^\mu \rho \tag{3.11}$$

$$\begin{aligned} R_{11} - R_{22} &\equiv r^{-1}(\mu_2 + f^{-1}f_2) + \tfrac{1}{2}f^{-2}(f_1^2 - f_2^2) + \tfrac{1}{2}r^{-2}f^2(w_2^2 - w_1^2) \\ &= 2r^{-2}[f(\psi_2^2 - \psi_1^2) + l(\phi_1^2 - \phi_2^2) + 2m(\phi_2\psi_2 - \phi_1\psi_1)] \end{aligned} \tag{3.12}$$

$$2R_{12} \equiv -r^{-1}(\mu_1 + f^{-1}f_1) + f^{-2}f_1f_2 - r^{-2}f^2 w_1 w_2 = 4r^{-2}[l\phi_1\phi_2 - f\psi_1\psi_2 - m(\phi_1\psi_2 + \phi_2\psi_1)] \tag{3.13}$$

$$\begin{aligned} R^4_4 - R^3_3 - 2wR^3_4 &\equiv f^{-1} e^{-\mu}[-\nabla^2 f + f^{-1}(f_1^2 + f_2^2) - r^{-2}f^3(w_1^2 + w_2^2)] \\ &= -2r^{-2} e^{-\mu}[f^{-1}(r^2 + m^2)(\phi_1^2 + \phi_2^2) + 2m(\phi_1\psi_1 + \phi_2\psi_2) + f(\psi_1^2 + \psi_2^2)] \\ &\quad + 8\pi\rho(u^3u_3 - u^4u_4 + 2wu^3u_4) \end{aligned} \tag{3.14}$$

$$2R^3_4 \equiv -r^{-2}\,\mathrm{e}^{-\mu}[f^2\nabla^{*2}w + 2f(f_1w_1+f_2w_2)]$$
$$= 4r^{-2}\,\mathrm{e}^{-\mu}[m(\phi_1^2+\phi_2^2)+f(\phi_1\psi_1+\phi_2\psi_2)] - 16\pi\rho u^3 u_4 \tag{3.15}$$

where I have put

$$w = mf^{-1} \tag{3.16}$$
$$\nabla^2 X = X_{11}+X_{22}+r^{-1}X_2 \tag{3.17}$$
$$\nabla^{*2} X = X_{11}+X_{22}-r^{-1}X_2. \tag{3.18}$$

The second set of Maxwell equations, (3.3), gives

$$f\nabla^{*2}\psi + m\nabla^{*2}\phi + f_1\psi_1 + f_2\psi_2 + m_1\phi_1 + m_2\phi_2 = 4\pi r^2\,\mathrm{e}^{\mu}J^3 \tag{3.19}$$
$$m\nabla^{*2}\psi - l\nabla^{*2}\phi + m_1\psi_1 + m_2\psi_2 - l_1\phi_1 - l_2\phi_2 = 4\pi r^2\,\mathrm{e}^{\mu}J^4. \tag{3.20}$$

The equations (3.11)–(3.15) and (3.19)–(3.20) agree with those given by Islam (1977; (20*a*)–(20*e*) and (21*a*)–(21*b*)) subject to the following remarks:

(i) Islam takes the opposite sign for R_{ik} expressed in terms of the Γ's;

(ii) in this paper u^3 and J^3, and in Islam's Ω and J^3, must be put to zero;

(iii) correction is required of the following trivial misprints in Islam's paper: in the last term of (20*d*), insert π; insert a minus sign on the right-hand side of (21*a*).

If ρ and J^i are put to zero, equations (3.11)–(3.15) and (3.19)–(3.20) reduce to those for the stationary, axially symmetric electromagnetic field, which have been studied by many authors, and of which many solutions are known. (For a summary see Kinnersley 1974.) *We shall henceforth assume, as in* § 2 *that* $\rho \neq 0$, $J^i \neq 0$ *except at isolated points.*

The structure of the equations is as follows. If we regard (3.12) and (3.13) as determining μ_2 and μ_1 in terms of the remaining quantities, the three equations (3.11)–(3.13) give two compatibility equations, arising from $\mu_{12}=\mu_{21}$, and from the condition that $\mu_{11}+\mu_{22}$ shall satisfy (3.11). If the compatibility equations, which, as we shall see, correspond to the equations of motion of the dust, are satisfied, μ can be determined up to an additive constant. The remaining equations (3.14), (3.15), (3.19) and (3.20) are four equations for four potentials f, m (or w), ϕ and ψ in terms of the four source functions ρ, J^3, J^4 and u^3/u^4. These functions are restricted by the equations of motion, and will be further restricted by assumptions we shall make ((4.6) and (5.1)).

4. The compatibility conditions

It will be convenient to introduce the angular velocity Ω by

$$u^3 = \Omega u^4, \tag{4.1}$$

whence, recalling that u^i is a time-like unit vector, we have

$$u^4 = (f-2m\Omega - l\Omega^2)^{-1/2}; \tag{4.2}$$

here, and throughout the paper, where a square root is shown, the positive value is to be taken.

Using (3.12) and (3.13) to form $\mu_{21}=\mu_{12}$, we find after a long calculation in which (3.14), (3.15), (3.19) and (3.20) are used that

$$\rho(f_1 - 2m_1\Omega - l_1\Omega^2)(u^4)^2 + 2(\psi_1 J^3 + \phi_1 J^4) = 0. \tag{4.3}$$

A similar long calculation, in which we substitute for μ_{11} and μ_{22} in (3.11), gives

$$\rho(f_2 - 2m_2\Omega - l_2\Omega^2)(u^4)^2 + 2(\psi_2 J^3 + \phi_2 J^4) = 0. \tag{4.4}$$

Equations (4.3) and (4.4) together form the equation of motion for the charged dust:

$$\rho u^i{}_{;k} u^k = J^k F_k{}^i. \tag{4.5}$$

Provided we ensure that (4.3) and (4.4) are satisfied we can ignore the field equations (3.11)–(3.13) except for the purpose of finding μ by quadratures. *Hence the complete solution depends essentially on the four quantities f, m, ϕ and ψ obtained by solving equations* (3.14), (3.15), (3.19) *and* (3.20).

I shall suppose from now on (as in § 2) that the dust carries the charge, so that the four-current arises from the charge density σ moving with velocity u^i:

$$J^i = \sigma u^i. \tag{4.6}$$

Then, putting

$$F = (f - 2m\Omega - l\Omega^2) \tag{4.7}$$

and using (4.3) and (4.4), *we obtain the equations of motion*

$$\rho(F^{1/2})_a + \sigma(\phi + \Omega\psi)_a + \Omega_a[\rho F^{-1/2}(m + l\Omega) - \sigma\psi] = 0 \tag{4.8}$$

where $a = 1, 2$. An equivalent equation was obtained by Islam (1977). *This equation is strictly comparable with the classical equation* (2.14), as I now show.

First recall that in static weak fields g_{44} is approximately $1 + 2V$, where V is the gravitational potential, supposed much less than 1; assume that this applies to f in (3.4) and suppose that the terms $2m\Omega$ and $l\Omega^2$ in (4.7) are much less than 1. Then, using (3.10), we find

$$F^{1/2} \sim 1 + V - m\Omega - \tfrac{1}{2}r^2\Omega^2.$$

Neglecting second-order terms like V^2, $Vm\Omega$, $m^2\Omega^2$, etc., equation (4.8) now gives

$$\rho(V - m\Omega - \tfrac{1}{2}r^2\Omega^2)_a + \sigma(\phi + \Omega\psi)_a + \Omega_a[\rho(m + r^2\Omega) - \sigma\psi] = 0,$$

where further small terms are neglected. *This is the same as the classical* (2.14) *if we neglect m.*

The presence of the function m is the main difference from the classical theory in the stationary axially symmetric case: m is related to the gravitational vector potential (Møller 1972) given in general by

$$\kappa_\alpha = g_{\alpha 4}(g_{44})^{-1/2} \qquad (\alpha = 1, 2, 3), \tag{4.9}$$

which has no analogue in Newtonian mechanics. In stationary, axially symmetric space–times κ_α reduces to one component, $-wf^{1/2}$ in our notation, which gives rise to one extra field equation (3.15), as well as to extra terms in the other equations.

Special cases of equation of motion (4.8) can be treated as for the Newtonian equation (2.14). However, it will be more convenient to deal with them after making, in the next section, a coordinate transformation allowed to us in the relativistic theory.

5. Rigidly rotating dust

Henceforth I shall suppose that the dust rotates rigidly, i.e. that

$$\Omega = \text{constant}. \tag{5.1}$$

It follows at once from (4.1) that the transformation $\bar{z} = z$, $\bar{r} = r$, $\bar{\theta} = \theta - \Omega t$, $\bar{t} = t$ makes

$$\bar{u}^3 = 0 \qquad \bar{\Omega} = 0$$

without affecting the form (3.4) of the metric, or the forms of A_i and J^i in (3.6). From now on we shall suppose this done, and put

$$u^3 = 0 \qquad \Omega = 0 \qquad u^4 = f^{-1/2} \qquad F = f, \tag{5.2}$$

the last two being a consequence of (4.2) and (4.7). The equation of motion (4.8) becomes

$$\rho(f^{1/2})_a + \sigma\phi_a = 0, \tag{5.3}$$

and we can apply to this a classification similar to that used on (2.14).

(*a*) *Case I.* $\phi \neq$ constant. This leads to

$$f = f(\phi) \tag{5.4}$$

$$\rho(f^{1/2})' + \sigma = 0 \tag{5.5}$$

where a prime means a differentiation with respect to ϕ.

(*b*) *Case I*(*a*). $f^{1/2}$ is a linear function of its argument so

$$f^{1/2} = n\phi + B \qquad \rho n + \sigma = 0 \tag{5.6}$$

where n is a constant. In this case the ratio of charge to mass density is constant. Since ϕ and ψ appear in the field equations only through their derivatives, they are undetermined up to arbitrary constants which may be used to eliminate B. Hence we shall write

$$f = n^2\phi^2. \tag{5.7}$$

(*c*) *Case II.* The Lorentz force vanishes: i.e. the term $(\phi + \Omega\psi)$ in (4.8) is constant, which in view of (5.2) reduces to

$$\phi = \text{constant}. \tag{5.8}$$

Equation (5.3) now requires f to be constant and by a scale charge of the time coordinate we may take

$$f = 1. \tag{5.9}$$

These results can conveniently be written as a theorem.

Theorem. Consider a space–time interior composed of axially symmetric, stationary, rigidly rotating charged dust in comoving coordinates and satisfying (3.6). Then

(*a*) if the Lorentz force does not vanish, σ/ρ and g_{44} are functions of the electric potential ϕ;

(*b*) in the special case in which σ/ρ is a constant, $-n$, then $g_{44} = n^2\phi^2$;

(*c*) if the Lorentz force vanishes, g_{44} and ϕ are constant.

All the parts of this theorem have been discovered before, or are implicit in previous work, but it seemed to me worthwhile to collect and prove them together. Part (*a*) can be obtained from the paper by Das and Kloster (1977), though they did not derive it because their interest was somewhat different; part (*b*) was proved by Islam (1978) and (*c*) by Banerjee *et al* (1976).

6. Solutions when the Lorentz force vanishes

Using (5.8) and (5.9), we find that the equations (3.13), (3.14), (3.19) and (3.20) reduce to

$$\nabla^{*2} w = 0 \tag{6.1a}$$

$$\nabla^{*2} \psi = 0 \tag{6.1b}$$

$$8\pi\rho + 2r^{-2}\,\mathrm{e}^{-\mu}(\psi_1^2 + \psi_2^2) = r^{-2}\,\mathrm{e}^{-\mu}(w_1^2 + w_2^2), \tag{6.2}$$

$$4\pi\sigma = r^{-2}\,\mathrm{e}^{-\mu}(w_1\psi_1 + w_2\psi_2). \tag{6.3}$$

μ can be obtained by integrating (3.12) and (3.13) when w and ψ have been chosen. Equations (6.1) can be reduced to Laplace equations by substitutions of the form

$$w = r\eta_2$$

which gives

$$\nabla^{*2} w \equiv r\frac{\partial}{\partial r}(\nabla^2 \eta) \tag{6.4}$$

so $\nabla^{*2} w = 0$ if η is a harmonic function†. A similar procedure may be used for ψ so we see that *our solution depends on two harmonic functions. It is the general solution for vanishing Lorentz force.* The solution of Islam (1977) is a special case obtained by taking w a constant multiple of ψ, and solutions of this type were also obtained by Kloster and Das (1977). For this special case σ/ρ is constant, but this is not generally so.

There is evidently an analogy between this class of solutions and that of case II in § 2. The correspondence can be seen between the equations for the magnetic potentials (6.1*b*) and (2.25), between the mass densities (6.2) and (2.23), and between the charge densities (6.3) and (2.24). There is no Newtonian analogue of the equation (6.1*a*) for the gravitational vector potential component. Equation (6.2) is interesting: the magnetic energy (second term on the left) is added to the mass density in balancing the rotation terms on the right.

As a simple example of the relativistic solution we can take the case in which both w and ψ are proportional to r^2:

$$w = k_1 r^2 \qquad \psi = k_2 r^2 \qquad (k_1,\, k_2 \text{ constant}).$$

These satisfy (6.1). The remaining quantities are found to be

$$\mu = (4k_2^2 - k_1^2)r^2$$

$$\rho = \pi^{-1}\,\mathrm{e}^{-\mu}(\tfrac{1}{2}k_1^2 - k_2^2)$$

$$\sigma = \pi^{-1}\,\mathrm{e}^{-\mu}k_1k_2.$$

This solution was first obtained by Som and Raychaudhuri (1968). It has cylindrical symmetry and only a longitudinal magnetic field; it corresponds with the classical solution generated by (2.26), but is somewhat simpler!

† To the harmonic function can be added an aribtrary function of z, but this produces no addition to w.

7. Solutions when the Lorentz force is not zero

In this case we know from (5.6) and (5.7) that f, the scalar gravitational potential, is a function of the electric scalar potential ϕ. As an ansatz to find a solution, we shall try making the gravitational vector potential (4.9) a function of the magnetic vector potential. In our case this means supposing

$$wf^{1/2}=g(\psi). \tag{7.1}$$

This does lead to a simplification for certain functions $f(\phi)$ and $g(\psi)$, as we shall see.

Using (5.2), (5.4) and (7.1) in (3.14), we obtain

$$\begin{aligned}8\pi e^{\mu}f\rho = f'\nabla^2\phi + (f''-f^{-1}f'^2+\tfrac{1}{4}r^{-2}g^2f'^2-2r^{-2}fg^2-2)(\phi_1^2+\phi_2^2)\\ -r^{-2}fg(f'\dot{g}+4f^{1/2})(\phi_1\psi_1+\phi_2\psi_2)+r^{-2}f^2(\dot{g}^2-2)(\psi_1^2+\psi_2^2)\end{aligned} \tag{7.2}$$

where a dot means $d/d\psi$.

Next we multiply (3.19) by m and substract (3.20) multiplied by f, use (5.4) and (7.1), and obtain

$$\begin{aligned}2\pi\, e^{\mu}f'\rho = \nabla^2\phi + (\tfrac{1}{2}r^{-2}g^2-f^{-1})f'(\phi_1^2+\phi_2^2)\\ -r^{-2}fg(\dot{g}-\tfrac{1}{2}f'f^{-1/2})(\phi_1\psi_1+\phi_2\psi_2)-r^{-2}f^{3/2}\dot{g}(\psi_1^2+\psi_2^2).\end{aligned} \tag{7.3}$$

We now multiply (7.2) by f' and (7.3) by $4f$ and subtract, so eliminating ρ and obtaining

$$\begin{aligned}0=(f'^2-4f)\nabla^2\phi+f'(f''-f^{-1}f'^2+\tfrac{1}{4}r^{-2}g^2f'^2+2-4r^{-2}fg^2)(\phi_1^2+\phi_2^2)\\ -r^{-2}fg(f'^2\dot{g}+4f'f^{1/2}-4\dot{g}f+2f'f^{1/2})(\phi_1\psi_1+\phi_2\psi_2)\\ +r^{-2}f^2(f'\dot{g}^2-2f'+4f^{1/2}\dot{g})(\psi_1^2+\psi_2^2).\end{aligned} \tag{7.4}$$

As we shall see shortly, a simplification will come from a special choice of f and g.

We carry out similar operations on (3.15), getting

$$\begin{aligned}f\dot{g}\nabla^{*2}\psi-\tfrac{1}{2}f'g\nabla^{*2}\phi+(4-\tfrac{1}{4}f^{-1}f'^2-\tfrac{1}{2}f'')g(\phi_1^2+\phi_2^2)\\ +(4f^{1/2}+f'\dot{g})(\phi_1\psi_1+\phi_2\psi_2)+f\ddot{g}(\psi_1^2+\psi_2^2)=0,\end{aligned} \tag{7.5}$$

and on (3.19) with the result

$$f\nabla^{*2}\psi+f^{1/2}g\nabla^{*2}\phi+\tfrac{1}{2}f^{-1/2}f'g(\phi_1^2+\phi_2^2)+(f'+f^{1/2}\dot{g})(\phi_1\psi_1+\phi_2\psi_2)=0. \tag{7.6}$$

We now eliminate $\nabla^{*2}\psi$ between (7.5) and (7.6), obtaining

$$\begin{aligned}(f^{1/2}\dot{g}+\tfrac{1}{2}f')g\nabla^{*2}\phi+(\tfrac{1}{2}f''-4+\tfrac{1}{4}f^{-1}f'^2+\tfrac{1}{2}f^{-1/2}f'\dot{g})g(\phi_1^2+\phi_2^2)\\ +f^{1/2}(\dot{g}^2-4)(\phi_1\psi_1+\phi_2\psi_2)-f\ddot{g}(\psi_1^2+\psi_2^2)=0.\end{aligned} \tag{7.7}$$

There are now two differential equations containing $\phi_{11}+\phi_{22}$, namely (7.4) and (7.7). We could proceed if the functions $f(\phi)$ and $g(\psi)$ could be chosen so that they reduced to the same equation, but this does not seem to be possible. However, we can by a suitable choice of f and g make (7.7) vanish identically, leaving (7.4) as the sole equation for ϕ. The appropriate choice is

$$f^{1/2}=-2\epsilon\phi \qquad g=2\epsilon\psi, \qquad \epsilon\phi=-|\phi|, \tag{7.8}$$

so that ϵ is $+1$ or -1 according as ϕ is negative or positive. We assume $\phi\neq 0$ to avoid a zero in f, which would lead to a singularity in u^i (see (5.2)).

Using (7.8) we find that (7.4) reduces to

$$\phi\nabla^2\phi-(\phi_1^2+\phi_2^2)=0 \tag{7.9}$$

which is equivalent to

$$\nabla^2(\ln\phi)=0. \tag{7.10}$$

The function m in the metric is given by

$$m=-4\phi\psi, \tag{7.11}$$

and ψ satisfies, because of (7.6) and (7.8),

$$\phi^2\nabla^{*2}\psi-\phi\psi\nabla^{*2}\phi-\psi(\phi_1^2+\phi_2^2)+\phi(\phi_1\psi_1+\phi_2\psi_2)=0. \tag{7.12}$$

It is convenient to reinstate $w=-\phi^{-1}\psi$; equation (7.12) is found to be equivalent to

$$\nabla^{*2}w+3\phi^{-1}(\phi_1w_1+\phi_2w_2)=0. \tag{7.13}$$

Now write this as

$$\frac{\partial}{\partial z}\left(\frac{w_1\phi^3}{r}\right)+\frac{\partial}{\partial r}\left(\frac{w_2\phi^3}{r}\right)=0,$$

showing that there exists a function $Y(z,r)$ such that

$$r^{-1}w_1\phi^3=Y_2 \qquad r^{-1}w_2\phi^3=-Y_1, \tag{7.14}$$

and, because $w_{12}=w_{21}$, satisfying

$$\nabla^2Y-3\phi^{-1}(\phi_1Y_1+\phi_2Y_2)=0. \tag{7.15}$$

Y is somewhat similar to the twist potential used in the solution of the stationary electrovacuum case. The mass density is given by either (7.2) or (7.3) in the form

$$\pi\rho+(4\phi)^{-2}(\phi_1^2+\phi_2^2)\,e^{-\mu}=\phi^{-2}(Y_1^2+Y_2^2)\,e^{-\mu}, \tag{7.16}$$

the physical interpretation of which being that the total density of energy (including the electromagnetic energy represented by the second term on the left) is balanced by the rotation terms on the right. The ratio of charge to mass density is

$$\sigma/\rho=2\epsilon \tag{7.17}$$

from (5.6) and (7.8). μ can be obtained from (3.12) and (3.13) and may be expressed in the form

$$(\mu+\ln f)_1=3r\phi^{-2}\phi_1\phi_2 \qquad (\mu+\ln f)_2=\tfrac{3}{2}r\phi^{-2}(\phi_2^2-\phi_1^2). \tag{7.18}$$

To sum up, a solution is obtained by choosing a harmonic function for $\ln\phi$ *and then solving the linear second-order partial differential equation* (7.13) *for* w, which is equivalent to (7.15) for Y. ψ is then obtained from $\psi=-w\phi$, f from (7.8), ρ and σ from (7.16) and (7.17) and μ from (7.18).

If in (7.15) we require that Y be a function of ϕ we obtain

$$Y=K\phi^3+K^*, \tag{7.19}$$

K and K^* being constants, of which the latter may be put zero because Y enters the solution only through its derivatives. From (7.14) we can obtain w by integrating

$$w_1=3rK\phi^{-1}\phi_2 \qquad w_2=-3rK\phi^{-1}\phi_1 \tag{7.20}$$

for any ϕ which satisfies (7.10). *This procedure gives a class of solutions of the problem depending on one harmonic function* ($\ln \phi$), *an arbitrary constant K, and a further arbitrary constant obtained on integrating* (7.20).

Using (7.20) we can obtain solutions regular near the origin of coordinates. For example, choosing

$$\ln \phi = C(2z^2 - r^2) \tag{7.21}$$

where C is a constant, we find by integrating (7.19)

$$w = -6Kr^2 z \tag{7.22}$$

ignoring an additive constant to avoid a singularity on the rotation axis. ψ, μ, ρ and σ are easily obtained and the solution is found to be non-singular for the finite part of space–time. Thus it could serve as an interior solution if a suitable exterior could be found.

The solution is not, of course, the only one corresponding to the harmonic function (7.21). Indeed we can, instead of assuming Y to be of form (7.19), write it as $Y = Z(z)R(r)$, and separate the variables in (7.15). In this way we obtain a more general solution, showing that the prescription of the electric field, as in (7.21), does not determine w and ψ.

Similarly, if one prescribes the cylindrically symmetric solution of (7.10), namely

$$\phi = Ar^n \tag{7.23}$$

where A, n are constants, there is much freedom in the choice of w. One may take w also to be a function of r only (Islam 1978), but any solution of (7.13) which, with (7.23), becomes

$$w_{11} + w_{22} + (3n-1)r^{-1}w_2 = 0$$

is allowable. The class of solutions (7.19) makes w a function of z only. However, all these cylindrical solutions are singular because, by (7.8), f vanishes or diverges on $r = 0$.

There is no close analogy with the corresponding classical case (case I(a) of § 2). Nor is it apparent why the particular charge to mass ratio $\sigma/\rho = \pm 2$ should give rise to a solution in the relativistic case. Presumably the very weak coupling of the electric to the magnetic (or rotation) field is connected with this special ratio.

8. Conclusion

The equations of motion for stationary, axially symmetric charged dust are similar in the classical and relativistic theories (equations (2.14) and (4.8)). The main difference is the presence in the latter theory of m, which corresponds to the gravitational vector potential. This is also apparent in the field equations: in the classical theory there are three potentials, V, ϕ and ψ; but in relativity there are four, f (corresponding to V), ϕ, ψ and m (or w).

The exact solutions derived in this paper apply to rigidly rotating dust. In both theories they conveniently divide into two classes, according as the Lorentz force does not or does vanish. In the second class the general solution in the classical case is generated by a solution of (2.25), and in the relativistic case it depends on two harmonic equations (6.1). There is a noticeable similarity between solutions in the two theories.

In the first class one can in the classical theory obtain a linear equation (2.20) for a function which will generate the solution. In the relativistic theory the situation is much more complicated, and only a limited class of solutions, depending on one harmonic equation (7.10) and two constants, was obtained. The class has charge/mass density ratio of ± 2, and it would be interesting to know why a simple solution is obtainable in this special case.

References

Banerjee A and Banerji S 1968 *J. Phys. A: Gen. Phys.* **1** 188
Banerjee A, Chakravarty N and Dutta Choudhury S B 1976 *Aust. J. Phys.* **29** 119
Das A and Kloster S 1977 *J. Math. Phys.* **18** 586
De U K and Raychaudhuri A K 1968 *Proc. R. Soc.* A **303** 97
Kinnersley W 1974 *General Relativity and Gravitation* ed. G Shaviv and J Rosen *Proc. 7th Int. Conf. (GR7)* (New York: Wiley) p 109
Kloster S and Das A 1977 *J. Math. Phys.* **18** 2191
Islam J N 1977 *Proc. R. Soc.* A **353** 523
—— 1978 *Proc. R. Soc.* A **362** 329
—— 1979 *Proc. R. Soc.* A **367** 271
Møller C 1972 *The Theory of Relativity* (Oxford: Clarendon) p 279
Raychaudhuri A K and De U K 1970 *J. Phys. A: Gen Phys.* **3** 263
Som M M and Raychaudhuri A K 1968 *Proc. R. Soc.* A **304** 8
Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland) p 309